

A MODIFICATION OF THE NEWTON'S COOLING LAW AND MPEMBA EFFECT

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Abstract

In this work we suggest a simple modification of the Newton's cooling law that can model Mpemba effect. We introduce, in the usual Newton's law, i.e. linear differential equation, an additional term proportional to the quadrate of the geometrical average value of the initial and latter difference between liquid and cooling thermostat (environment) temperature. It, after simple transformations, yields usual Newton's linear differential equation but with modified cooling parameter. This modified cooling parameter represents sum of the usual cooling parameter and an additional term directly proportional to the difference between initial temperature of the liquid and cooling thermostat temperature. Corresponding solution of the modified Newton's cooling equation, i.e. temperature decrease during time, has an additional exponential term with negative argument proportional to mentioned difference between initial temperature of the liquid and temperature of the cooling thermostat. (It can be observed that appearance of such "hysteresis" or "memory" terms can imply some circulation effects by liquid flow but it goes over basic intention of this work.) It admits that liquid with higher initial temperature, after a characteristic time interval, cools faster than the same liquid with lower initial temperature, in full agreement with experimental data on the Mpemba effect.

In this work we shall suggest a simple modification of the Newton's cooling law [1] that can model Mpemba effect [2]-[5]. We shall introduce, in the usual Newton's law, i.e. linear differential equation, an additional term proportional to the quadrate of the geometrical average value of the initial and latter difference between liquid and cooling thermostat (environment) temperature. It, after simple transformations, yields usual Newton's linear differential equation but with modified cooling parameter. This modified cooling parameter represents sum of the usual cooling parameter and an additional term directly proportional to the difference between initial temperature of the liquid and cooling thermostat temperature. Corresponding solution of the modified Newton's cooling equation, i.e. temperature decrease during time, has an additional exponential term with

negative argument proportional to mentioned difference between initial temperature of the liquid and temperature of the cooling thermostat. (It can be observed that appearance of such "hysteresis" or "memory" terms can imply some circulation effects by liquid flow but it goes over basic intention of this work.) It admits that liquid with higher initial temperature, after a characteristic time interval, cools faster than the same liquid with lower initial temperature, in full agreement with experimental data on the Mpemba effect.

As it is well-known [1] usual Newton's law of cooling represents the following simple linear differential equation

$$\frac{dT}{dt} = -a(T - T_e) \quad (1)$$

with simple solution

$$T = T_e + (T_0 - T_e) \exp[-at]. \quad (2)$$

Here T represents the temperature of a liquid with initial value T_0 , T_e - constant, smaller than T , temperature of a cooling wall (thermostat or environment) in the contact with liquid, while a represents a cooling parameter that simplifiedly expresses thermodynamics of the cooling process. It is supposed implicitly that cooling parameter is independent of the initial temperature of the liquid as well as of the cooling wall temperature.

Consider now described by (1) or (2) thermodynamical interaction between liquid and cooling wall in two different cases, precisely for two different initial temperatures of the liquid, first one T_{10} and second one T_{20} so that T_{10} is larger than T_{20} , i.e.

$$T_{10} > T_{20}. \quad (3)$$

Then (2), (3) imply

$$\frac{T_1 - T_e}{T_2 - T_e} = \frac{T_{10} - T_e}{T_{20} - T_e} \quad (4)$$

where T_1 and T_2 represents the temperature of the liquid in the first case and second case in some (finite) later time moment t . It means that temperature of the liquid in the first case will be larger than the temperature of the liquid in the second case in any (finite) later time moment.

Previous conclusion implies unambiguously that usual Newton's cooling law (1) cannot model experimental data of the Mpemba effect [2]-[5] according to which, simply speaking, hot liquid freezes faster than cold liquid.

Also, it is well known [1] that there are many different non-linear modifications of the Newton cooling law. For example there is the following modification

$$\frac{dT}{dt} = -a(T - T_e)^n \quad (5)$$

for $1.3 < n < 1.6$ in the Taylor's model and for $n = \frac{5}{4}$ in the Dulong-Petit model. Or, there is so-called Newton - Stefan modification of the usual Newton's cooling law

$$\frac{dT}{dt} = -a(T - T_e) - b(T - T_e)^2 - c(T - T_e)^3 - d(T - T_e)^4. \quad (6)$$

where cooling parameters a , b and c are dependent of T_e .

We shall suggest a simple, quasi-quadratic modification of the usual Newton's cooling law

$$\frac{dT}{dt} = -a(T - T_e) - b < T - T_e >^2 \quad (7)$$

where

$$\langle T - T_e \rangle = ((T - T_e)(T_0 - T_e))^{\frac{1}{2}} \quad (8)$$

represents the geometric average value of the initial and latter difference between liquid and cooling wall temperature.

Introduction of (8) in (7) yields

$$\frac{dT}{dt} = -(a + b(T_0 - T_e))(T - T_e) \quad (9)$$

representing a linear differential equation similar to usual Newton's cooling law (1). However, in distinction to usual Newton's cooling law, (9) holds cooling parameter $(a + b(T_0 - T_e))$ dependent of the initial difference between liquid and cooling wall.

As it is not hard to see equation (9) holds the simple solution

$$T = T_e + (T_0 - T_e) \exp[-(a + b(T_0 - T_e))t]. \quad (10)$$

It can be observed that in the limit of small $(T_0 - T_e)$, i.e. for $(T_0 - T_e) \ll \frac{a}{b}$, modified Newton's cooling law, i.e. equation (9) and its solution (10) turn out usual Newton's cooling law, i.e. in the equation (1) and its solution (2).

Consider now described by (9) or (10) thermodynamical interaction between liquid and cooling wall in two different cases, precisely for two different initial temperatures of the liquid, first one T_{10} and second one T_{20} so that T_{10} is larger than T_{20} , i.e.

$$T_{10} > T_{20}. \quad (11)$$

Then, according to (9)-(11), between temperature of the liquid in the first case T_1 and temperature of the liquid in the second case T_2 , there is the following relation

$$\frac{T_1 - T_e}{T_2 - T_e} = \frac{T_{10} - T_e}{T_{20} - T_e} \exp[-b(T_{10} - T_{20})t]. \quad (12)$$

It represents a time decreasing function that is, according to (11), initially larger than 1 but that after a characteristic time moment

$$\tau = \frac{1}{b(T_{10} - T_{20})} \ln \left[\frac{T_{10} - T_e}{T_{20} - T_e} \right] \quad (13)$$

becomes smaller than 1. In other words, after characteristic time moment τ , T_1 becomes smaller than T_2 even if T_{10} is larger than T_{20} .

As it is not hard to see, all this can be used as a simple model for the Mpemba effect [2]-[5].

It can be observed that average temperature difference $\langle T - T_e \rangle$ (8) representing geometrical average value of the initial and later temperature difference does not represent unique form of the average temperature difference that introduced in (7) can model the Mpemba effect. For example, as it is not hard to see (but which will not be analyzed here explicitly), arithmetical average value of the initial and later temperature difference introduced in (7) can model the Mpemba effect too. But, in this case, expression (7) represents the differential equation with constant, linear and really existing quadratic term depending of $T - T_e$ as the variable and $T_0 - T_e$ as the parameter (which implies that solution of such equation decreases during time faster than (10)).

In conclusion it can be repeated and pointed out the following. In this work we suggest a simple modification of the Newton's cooling law that can model Mpemba effect. We introduce,

in the usual Newton's law, i.e. linear differential equation, an additional term proportional to the quadrate of the geometrical average value of the initial and latter difference between liquid and cooling thermostat (environment) temperature. It, after simple transformations, yields usual Newton's linear differential equation but with modified cooling parameter. This modified cooling parameter represents sum of the usual cooling parameter and an additional term directly proportional to the difference between initial temperature of the liquid and cooling thermostat temperature. Corresponding solution of the modified Newton's cooling equation, i.e. temperature decrease during time, has an additional exponential term with negative argument proportional to mentioned difference between initial temperature of the liquid and temperature of the cooling thermostat. (It can be observed that appearance of such "hysteresis" or "memory" terms can imply some circulation effects by liquid flow but it goes over basic intention of this work.) It admits that liquid with higher initial temperature, after a characteristic time interval, cools faster than the same liquid with lower initial temperature, in full agreement with experimental data on the Mpemba effect.

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